

**CORRIGENDUM TO 'TIME-INCONSISTENT MEAN-FIELD OPTIMAL STOPPING: A  
LIMIT APPROACH'**

BOUALEM DJEHICHE AND MATTIA MARTINI

A CORRECTION OF THE PROOF OF THEOREM 3.3

We provide a correction of proof of Theorem 3.3 in [DM23]. This result plays a key role in the proof of the main result of [DM23], Theorem 3.4. In the proof of Theorem 3.3, we used a well known law of large numbers whose statement in Eq. (3.24) is unfortunately wrong.

First, we recall that the family of interacting Snell envelopes  $\{Y^{i,n}\}_{i=1}^n$  and the family of finite horizon stopping problems  $\{Y^i\}_{i \geq 1}$  are defined by (2.4) and (2.5) in [DM23]. The function  $h$  and sequence  $\{\bar{c}^i\}_{i \geq 1}$  satisfy Assumption 2.1 in [DM23] (in what follows we will simply say that Assumption 2.1 holds). For the new proof, we need to introduce a (fixed) sequence of random variables  $\{\alpha_j\}_{j \geq 1}$  which are all independent of  $\{\mathbb{F}^i\}_{i \geq 1}$  and for which

$$(1.1) \quad \mathbb{E}[\alpha_j] = 1 - 2^{-j}, \quad \text{Var}(\alpha_j) \leq a^j, \quad j \geq 1, \quad |\mathbb{E}[\alpha_j \alpha_k]| \leq a^{|j-k|}, \quad j, k \geq 1,$$

for a given  $a \in (0, 1)$ . We note that

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (\mathbb{E}[\alpha_j] - 1)^2, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \text{Var}(\alpha_j) = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n a^j = 0.$$

The next lemma is the main ingredient in the new proof of Theorem 3.3.

**Lemma 1.1.** *Let Assumption 2.1 hold. Then the following law of large numbers (LLN) holds*

$$(1.3) \quad \lim_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \mathbb{E} \left[ \text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n} \sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] = 0.$$

Moreover,

$$(1.4) \quad \lim_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \mathbb{E} \left[ \text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n} \sum_{j=1}^n (1 - \alpha_j) Y_\tau^j \right)^2 \right] = 0.$$

*Proof.* Due to (1.1), the limit (1.4) is straightforward. To show (1.3), we note that since

$$\frac{1}{n} \sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) = \frac{1}{n} (\alpha_i Y_\tau^i - \mathbb{E}[\alpha_i Y_\tau^i]) + \frac{n-1}{n} \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]),$$

by Dominated Convergence it suffices to show

$$(1.5) \quad \lim_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \mathbb{E} \left[ \text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] = 0.$$

By the properties of the essential supremum, for each  $n \geq 2$ , there exists a sequence  $\{\tau_m^n\}_{m \geq 1}$  from  $\mathcal{T}_0^i$  such that

$$\text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 = \lim_{m \rightarrow \infty} \left( \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\alpha_j Y_{\tau_m^n}^j - \mathbb{E}[\alpha_j Y_{\tau_m^n}^j]) \right)^2 \quad \text{a.s.}$$

and by Dominated Convergence, we have

$$\begin{aligned} \mathbb{E} \left[ \text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n-1} \sum_{j \neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] &= \lim_{m \rightarrow \infty} \mathbb{E} \left[ \left( \frac{1}{n-1} \sum_{j \neq i}^n (\alpha_j Y_{\tau_m^n}^j - \mathbb{E}[\alpha_j Y_{\tau_m^n}^j]) \right)^2 \right] \\ &\leq \sup_{\tau \in \mathcal{T}_0^i} \mathbb{E} \left[ \left( \frac{1}{n-1} \sum_{j \neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right]. \end{aligned}$$

Now, by direct calculations it holds that, for every  $\tau \in \mathcal{T}_0^i$  and every  $\ell, j \neq i$ ,

$$\begin{aligned} \mathbb{E}[Y_\tau^\ell] &= \mathbb{E}[Y_\tau^j] = \mathbb{E}[\mathbb{E}[Y_s^1]_{s=\tau}], & \mathbb{E}[Y_\tau^\ell Y_\tau^j] &= \mathbb{E}[(\mathbb{E}[Y_s^1])^2]_{s=\tau} \geq 0, \\ \text{cov}(\alpha_j Y_\tau^j, \alpha_k Y_\tau^k) &= \text{cov}(\alpha_j, \alpha_k) \mathbb{E}[Y_\tau^j Y_\tau^k] + \mathbb{E}[\alpha_j] \mathbb{E}[\alpha_k] \text{cov}(Y_\tau^j, Y_\tau^k). \end{aligned}$$

Thus, the independence of  $(\alpha_j, \alpha_k)$  from  $(Y_\tau^j, Y_\tau^k)$  entails

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{1}{n-1} \sum_{j=1, j \neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] &= \frac{1}{(n-1)^2} \sum_{j,k, j,k \neq i}^n \text{cov}(\alpha_j Y_\tau^j, \alpha_k Y_\tau^k) \\ &\leq \left( \frac{1}{n-1} + \frac{n^2}{(n-1)^2} \frac{2}{n} \sum_{m=1}^n a^m \right) \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^1|^2 \right]. \end{aligned}$$

This bound, being uniform in  $\tau$  and in  $i, 1 \leq i \leq n$ , yields (1.3) due to (1.2).  $\square$

We will now state a new and correct version of [DM23, Theorem 3.3] and sketch its proof. To this end we need to substitute the smallness condition  $\gamma_1^2 + \gamma_2^2 < \frac{1}{16}$  with a new one.

**Theorem 1.2.** *Let Assumptions 2.1 hold and let us assume that  $\gamma_1$  and  $\gamma_2$  satisfy the new condition*

$$(1.6) \quad \gamma_1^2 + 3\gamma_2^2 < \frac{1}{28}.$$

Then, it holds that

$$(1.7) \quad \lim_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^{i,n} - Y_t^i|^2 \right] = 0.$$

*Proof.* As in the proof of [DM23, Theorem 3.3], we have that for any  $t \leq T$ ,

$$\begin{aligned} |Y_t^{i,n} - Y_t^i| &\leq \mathbb{E} \left[ \gamma_1 \sup_{s \in [0, T]} |Y_s^{i,n} - Y_s^i| + \frac{\gamma_2}{n} \sum_{j=1}^n \sup_{s \in [0, T]} |Y_s^{j,n} - Y_s^j| \right. \\ &\quad \left. + \frac{\gamma_2}{n} \sum_{j=1}^n \mathbb{E}[|\alpha_j|] \mathbb{E}[\sup_{s \in [0, T]} |Y_s^{j,n} - Y_s^j|] + \frac{\gamma_2}{n} \sum_{j=1}^n \mathbb{E}[|\alpha_j|] \mathbb{E} \left[ \sup_{s \in [0, T]} |Y_s^{j,n} - Y_s^j| \mid \mathcal{F}_t^i \right] \right. \\ &\quad \left. + \gamma_2 \mathbb{E}[\text{ess sup}_{\tau \in \mathcal{T}_0^i} |\frac{1}{n} \sum_{j=1}^n (\alpha_j - 1) Y_\tau^j| \mid \mathcal{F}_t^i] + \gamma_2 \mathbb{E}[\text{ess sup}_{\tau \in \mathcal{T}_0^i} |\frac{1}{n} \sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])| \mid \mathcal{F}_t^i] \right. \\ &\quad \left. + \gamma_2 |\frac{1}{n} \sum_{j=1}^n (\mathbb{E}[\alpha_j] - 1)| \mathbb{E}[\sup_{s \in [0, T]} |Y_s^1|] \right]. \end{aligned} \quad (1.8)$$

Set

$$\begin{aligned} \Lambda_n &:= 28\gamma_2^2 \sup_{1 \leq i \leq n} \mathbb{E} \left[ \text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n} \sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] \\ &\quad + 28\gamma_2^2 \sup_{1 \leq i \leq n} \mathbb{E} \left[ \text{ess sup}_{\tau \in \mathcal{T}_0^i} \left( \frac{1}{n} \sum_{j=1}^n (\alpha_j - 1) Y_\tau^j \right)^2 \right] + 28\gamma_2^2 \left| \frac{1}{n} \sum_{j=1}^n (\mathbb{E}[\alpha_j] - 1) \right|^2 \mathbb{E} \left[ \sup_{s \in [0, T]} |Y_s^1| \right]^2. \end{aligned}$$

Since  $\mathbb{E}[\alpha_j^2] \leq 1$ , in view of the exchangeability of  $\{Y^{j,n}, Y^j\}_{j=1}^n$ , the Cauchy-Schwarz inequality and Doob's inequality, if we set  $C := (1 - 28(\gamma_1^2 + 3\gamma_2^2))^{-1}$ , by (1.1) and Lemma 1.1 we have

$$(1.9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^{i,n} - Y_t^i|^2 \right] \leq C \lim_{n \rightarrow \infty} \Lambda_n = 0.$$

But, again due to the exchangeability of  $\{Y^{j,n}, Y^j\}_{j=1}^n$ , it holds that

$$(1.10) \quad \sup_{1 \leq i \leq n} \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^{j,n} - Y_t^j|^2 \right] = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^{j,n} - Y_t^j|^2 \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where the limit follows from (1.9). Now, from (1.8), we get

$$(1.11) \quad \begin{aligned} \frac{1}{28\gamma_2^2} (1 - 28\gamma_1^2) \sup_{1 \leq i \leq n} \mathbb{E}[\sup_{t \in [0, T]} |Y_t^{i,n} - Y_t^i|^2] &\leq \sup_{1 \leq i \leq n} \frac{1}{n} \sum_{j=1}^n \mathbb{E}[\sup_{t \in [0, T]} |Y_t^{j,n} - Y_t^j|^2] \\ &\quad + 2 \frac{1}{n} \sum_{j=1}^n \mathbb{E}[\sup_{t \in [0, T]} |Y_t^{j,n} - Y_t^j|^2] + \frac{1}{28\gamma_2^2} \Lambda_n. \end{aligned}$$

Finally, since (1.6) entails  $28\gamma_1^2 < 1$  and in view of (1.3), (1.9) and (1.10), we obtain

$$(1.12) \quad \lim_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^{i, n} - Y_t^i|^2 \right] = 0.$$

□

## REFERENCES

- [DM23] B. Djehiche and M. Martini, *Time-inconsistent mean-field optimal stopping: A limit approach*, Journal of Mathematical Analysis and Applications **528** (2023), no. 1, 127582.

DEPARTMENT OF MATHEMATICS, KTH ROYAL INSTITUTE OF TECHNOLOGY, 100 44, STOCKHOLM, SWEDEN  
Email address: boualem@kth.se

UNIVERSITÉ CÔTE D'AZUR, CNRS, LABORATOIRE J. A. DIEUDONNÉ, 06108 NICE, FRANCE  
Email address: mattia.martini@univ-cotedazur.fr